PERGAMON

International Journal of Solids and Structures 36 (1999) 4269–4287

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

Modelling unilateral damage effect in strongly anisotropic materials by the introduction of the loading mode in damage mechanics

A. Thionnet[a,b,*], J. Renard[a]

[a] École Nationale Supérieure des Mines de Paris, Centre des Matériaux P. M. Fourt, BP 87-91003 Evry Cedex, France
[b] Université de Bourgogne, Laboratoire de Mécanique, BP 400-21011 Dijon Cedex, France

Received 12 November 1997; in revised form 2 June 1998

Abstract

Damage weakens the mechanical characteristics of materials. But this weakening can disappear if the cracks close again: this is called the unilateral effect of damage. We propose a model of this phenomenon using damage mechanics in the case of a diffuse network of identical microcracks. The microcracks' state is defined with two internal state variables. These two variables are control parameters of the geometry of the microcracks. They also define the loading mode in damage mechanics as in fracture mechanics. In order to limit the anisotropy induced by the microcracks, hence by the loading, we suppose that the geometry of damage spreads into preferential directions. Therefore, this model is essentially applied to materials with a strong anisotropy where the defects follow the constituents of the material. An application is given for composite laminates.

1. Introduction

In the case of compression loading, the cracks in a material induce particular effects on its behaviour. Since oriented cracks perpendicular to the direction of compressive loading close again, the rigidity of the damaged material can be restored. This is explained by the disappearance of the effect of damage from traction to compression: this is defined as the passive/active damage or as the unilateral effect of the damage. In the case where the cracks are not perpendicular to the compression loading, the crack lips are either opened or closed when sliding occurs. In both cases, the behaviour of the material is affected. In this paper we propose a model of this phenomenon in the case where the damage is a diffuse network of identical microcracks. We show that a definition

* Corresponding author. Tel.: 0033 0160763031; fax: 0033 0160763150; e-mail: jacques.renard@mat.ensmp.fr

0020-7683/99/$ - see front matter
PII: S0020-7683(98)00192-9

of the loading mode in damage mechanics, as in fracture mechanics, allows to take into account the unilateral effect of damage.

2. Bibliography and description of the model

In the case of isotropic materials, several models exhibit important incompatibilities with the principles of the mechanics of a continuous medium and of damage mechanics. Based on scalar (La Borderie et al., 1989), on vector (Krajcinovic and Fonseka, 1981) or on tensor (Ramtani, 1990) descriptions, they do not take into account the anisotropy induced by the damage and simultaneously give a continuous stress/strain relationship. However, by the use of many assumptions, other approaches succeed in meeting all the requirements. We can mention the work of Chaboche (1992), but in this case, since the direction of the damage depends on the loading, it is necessary to define the 'principal directions of the damage'. The problem is that the arbitrary definition of these directions leads to different results. For the same materials, we can also mention the works of Andrieux (1983) and for composite materials the works of Allen et al. (1986) and Lee et al. (1989) which are based on a microscopic description of the default. Although these approaches are very complete and fine, they seem difficult to apply because the needed identifications are difficult to achieve. But, they are used here to give a qualitative physical support to the proposed model.

In a general way, to simulate the unilateral effect of the damage, all models distinguish the traction and the compression domains. By using two scalar state variables, our approach avoids this separation which creates discontinuities. These two variables define the loading mode for a diffuse network of identical microcracks in damage mechanics, in the same way as for macroscopic cracks in fracture mechanics. They are also indicators of the geometry of the microcrack lips. This is the reason why the proposed model:

- uses a single state function the classic form of which always induces the symmetry of the stress tensor;
- takes into account the unilateral condition of damage;
- leads to a continuous behaviour tensor the transition from traction to compression path.

Finally, it is important to point out that the effects of damage modify all the components of the behaviour tensor (Thionnet and Renard, 1993). A dual formulation can be found in the works of Aussedat et al. (1995).

3. Mechanical framework, hypothesis and field of application

The main difficulty in modelling the damage is linked to the fact that the induced anisotropy depends on the loading direction. Here, we partially avoid this difficulty by supposing that the geometry of damage spreads into preferential directions. It is the case of composites with strong anisotropy where the defects follow the direction of the constituents of the material. Thus, while considering the fact that the loading acts on the direction of the damage, we limit its effect by taking a material with an initial anisotropy which evolves weakly.

For this reason, we illustrate our study by the analysis and the modelling of the microcracking phenomenon in laminate composite materials. The concept of stacking disoriented plies to form a laminated structure allows us to develop a model for the unidirectional ply (mesoscopic scale), only with the help of local information. A multi-scale procedure between the laminate (macroscopic scale) and the ply allows us to calculate the mechanical response of any laminate subjected to any multiaxial loading. We suppose the unidirectional ply built from an initially orthotropic material. We show that the particular geometry of the damage, implies that the material becomes only monoclinic.

The multiple microcracks which are supposed to be identical in a damaged ply, lead us to use damage mechanics. Thus, we substitute the damaged material by an equivalent one made of an homogeneous material. It is important to note that the damage has to homogeneous in order to apply our modelling. A state corresponding to little damage is a limit of the model. Moreover, we suppose that:

- The microcracks are opened or there is no friction in the case of contact between their surfaces.
- The appearance of the microcracks does not induce residual stresses which keep energy within the material. This seems to be a difficult hypothesis to justify, but, in the case of the considered materials, the analysis of their responses for cyclic loading shows that the irreversible strains, which prove the existence of residual stresses, are very weak.
- The appearance of the microcracks and their quasi-complete propagation are simultaneous. As a consequence, in the present work, we are interested in the multiplication of the microcracks and not in their propagation.

The local Cartesian coordinate system of the ply is (O, e_1, e_2, e_3) (Fig. 1a): the (Oe_1) axis along the fibre direction, (Oe_3) along the thickness direction and thus (e_1Oe_2) is in the plane of the ply. In order to simplify the model, we use the hypothesis of plane stresses. This is the case of thin structures in the direction of their thickness. This simplification does not change the generality of the study.

To obtain thermodynamically admissible evolutions, the local state hypothesis is considered. Its main purpose is to postulate that a given number of variables, called the state variables, is sufficient to describe the representative elementary volume of the material equivalent to the damaged material. The evolution of these variables must satisfy the second principle of thermodynamics.

To build our model and to explain the different choices made during the study, we use microscopic considerations. However, our purpose is not to build a microscopic model (at the scale of the fibres and the matrix) but a mesoscopic model. Also, the microscopic considerations although essential are only qualitative.

The experimental microscopic observations of Purslow (1981, 1986) explain that the microcracks which, at the scale of the ply, seem to follow the fibres cannot in fact be modelled in a fixed plane. Further, the works of Andrieux (1983) show how internal variables appear in the mesoscopic state function to account for the presence of the microcracks and the discontinuities that they induce.

Concerning tensor notation, Voigt' notation is chosen. Thus:

- the plane components of the stress (σ) and strain (ε) tensors are respectively written σ_i and ε_i where $i = 1, 2, 6$;
- the plane components of a fourth order tensor t of orthotropic behaviour are written t_{ij} where $i = 1, 2, 6$

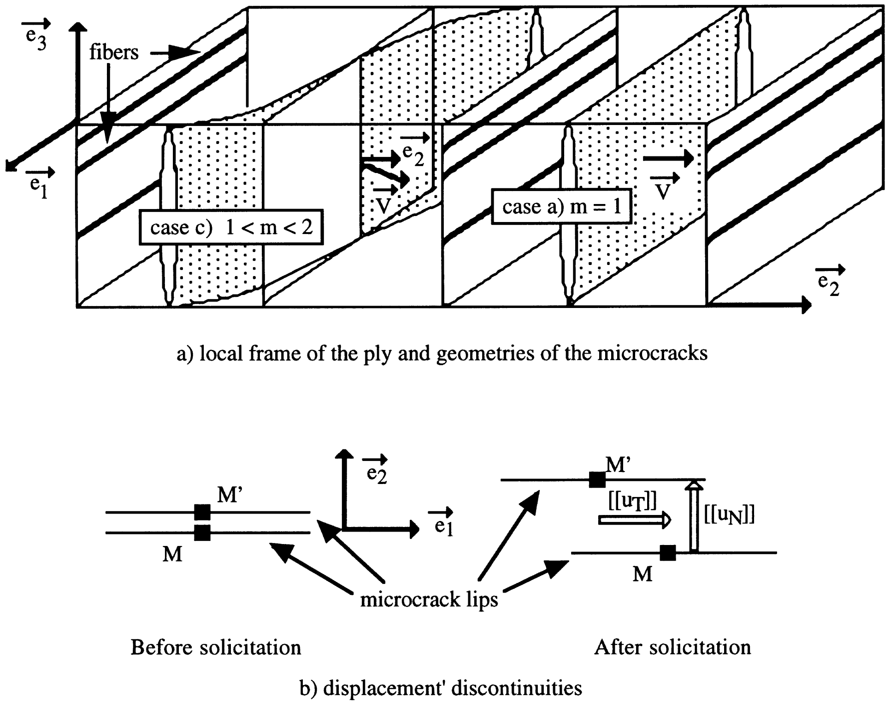

Fig. 1. Geometries of the microcracks.

4. Physical analysis of the damage: vectorial description of the evolutive geometry of microcracks

Geometrically, the microcracks occur all along the height of the cracked ply in a plane which contains the (Oe_3) and (Oe_1) axis, whatever the direction of solicitation. This last property mainly characterizes the materials showing strong anisotropy contrary to isotropic materials, the damage of them is guided by the applied loading.

In previous work (Thionnet and Renard, 1993), based on Talreja's works (Talreja, 1985), the microcracks were vectorially described by a vector perpendicular to their plane. Moreover, this direction was assumed to be fixed and parallel to the (Oe_2) axis. Indeed, we supposed that the microcracks were geometrically located in an average plane which contains the (Oe_3) axis of the thickness and the (Oe_1) axis of the fibres whatever the solicitation was.

However, a closer description of the phenomenon shows that the microcracks behave differently according to the submitted loadings. Thus, their fracture surfaces can be (Fig. 1a):

- case (a) opened (mode 1). In this case, the microcracks are in the (e_1Oe_3) plane and therefore the normal vector to the microcracks is colinear with the (Oe_2) axis as well as the displacement vector of their lips;
- case (b) slid one over another (mode 2);
- case (c) in a mixed configuration of the two preceeding possibilities. In this case, the microcracks are in an intermediate state between mode 1 and mode 2. Purslow's observations (Purslow, 1981) (Fig. 2) on a 90° carbon/epoxy specimen show, during shear loading, the existence of multiple

microcracks within the matrix which are not in the (e_1Oe_3) plane. Then, the normal vector to the microcracks has two non-zero components as well as the displacement vector of their lips.

Thus, the information arising from cases a, b and c explains why the microcracks have to be taken into account at the mesoscopic scale with a non-fixed direction.

Then, considering a volume element dV of material with F plane cracks. We can characterize each crack (k) by the following vector (in the local frame of the ply):

$$\overrightarrow{v^{(k)}} = D(l^{(k)}, L) \begin{pmatrix} v_{\mathrm{T}}^{(k)} = \int_{\text{crack}} [[u_{\mathrm{T}}]] \\ v_{\mathrm{N}}^{(k)} = \int_{\text{crack}} [[u_{\mathrm{N}}]] \\ 0 \end{pmatrix} \tag{1}$$

where: $[[u_{\mathrm{T}}]]$ and $[[u_{\mathrm{N}}]]$ are the tangential and normal displacement discontinuities on the micro-crack lips (Fig. 1b); D is a scalar function; $l^{(k)}$ denotes a characteristic size of crack k (length, area, ...); L denotes a characteristic size of the volume element (length, area, ...). Among these F vectors, it is natural to consider the case where several are combined. The cracks are then classified in N subfamilies as follows:

- n_1 cracks characterized by the vector $\overrightarrow{v_{(1)}}$;
- $n_{(i)}$ cracks characterized by the vector $\overrightarrow{v_{(i)}}$;
- $n_{(n)}$ cracks characterized by the vector $\overrightarrow{v_{(n)}}$.

Then the damage vector:

$$\overrightarrow{V_{(i)}} = \frac{1}{n_{(i)}} \sum_{i=1}^{n_{(i)}} \overrightarrow{v_{(i)}} \tag{2}$$

is associated with each subfamily i.

In the following we consider a single family of cracks characterized by the vector $\vec{V}$. This vector initially written $\vec{V} = (0, V_{\mathrm{N}}, 0)$ in the local frame of the ply in previous works (Thionnet and Renard, 1993), is now able to evolve in such a way to have two non-zero components: $\vec{V} = (V_{\mathrm{T}}, V_{\mathrm{N}}, 0)$.

5. Modelling

5.1. *Definition of the state variables*

We choose to write the thermodynamical state function in terms of strains. The strain tensor ε is therefore taken as a state variable. Its associated variable is σ. Concerning the damage, we choose $\alpha = e \cdot d$ where d is the microcrack density and e the thickness of the cracked ply. Former studies (Renard et al., 1993) based on an analysis of experimental or theoretical results (Reifsnider, 1977; Highsmith and Reifsnider, 1982; Lee et al., 1989) have shown that this parameter accounts well for the effects of the damage in composite laminates. Its associated variable is denoted as A.

The mesoscopic strain and the damage variable are not enough to characterize the damaged material. Indeed, it is necessary to introduce supplementary internal state variables describing tangential $[[u_T]]$ and normal $[[u_N]]$ displacement discontinuities induced by microcracks. Leguillon et al. (1982) show that a finite number of variables is enough to characterize the cracked material. On this basis, we define two independent internal mesoscopic variables characterizing the geometry of the microcrack lips: m called mode and r called radius. The associated variables respectively with m and r are noted M and R. Thus:

State variables	*Associated variables*
ε	σ
α	A
m	M
r	R

We will see the exact definition of the quantities m and r in the next section.

Before that, it is important to make the following remark. The vector $\vec{V}$ characterizing the microcracks network is used as an intermediate variable which help to write of the state function and not as a state variable. Thus, the components V_N and V_T depend on the internal state variables: $V_N = V_N(\alpha, m, r)$ and $V_T = V_N(\alpha, m, r)$. However, the value of the damage variable should not affect explicitly the direction of the damage vector but only the behaviour of the material. Therefore it is necessary (and sufficient) to take V_N and V_T with the following form:

$$V_N(\alpha, m, r) = f(\alpha)U_N(m, r) \quad \text{and} \quad V_T(\alpha, m, r) = f(\alpha)U_T(m, r). \tag{3}$$

5.2. A definition of the loading mode in damage mechanics

To define m and r two solutions are possible:

- either we know the discontinuities $[[u_N]]$ and $[[u_T]]$, i.e. we have access to a description and an identification at the microscopic level. Then, we are able to calculate m and r by a micro-meso procedure. This is for example the case when we are in the framework of infinite media (Andrieux, 1983);
- or we do not know the discontinuities $[[u_N]]$ and $[[u_T]]$, i.e. we do not have access to a description and an identification at the microscopic level. Then, we are not able to calculate m and r by a micro–meso procedure.

As we suppose in the second case, definitions of m and r have to be done at the mesoscopic level.

We define m as a scalar that describes, in the local frame of the ply and at the mesoscopic level, the opening mode of the microcracks. It represents for damage mechanics, the known notion of loading mode in fracture mechanics. In reference to the usual language, this notion should be:

- equal to 1 in the case where the loading on the microcracks network is such that the discontinuity of the tangential displacement is zero (mode 1);
- equal to 2 in the case where the loading on the microcracks network is such that the discontinuity of the normal displacement is zero (mode 2);
- equal to all intermediate values in mixed cases.

We define r as a scalar that describes, in the local frame of the ply and at the mesoscopic level, the space between the microcracks lips.

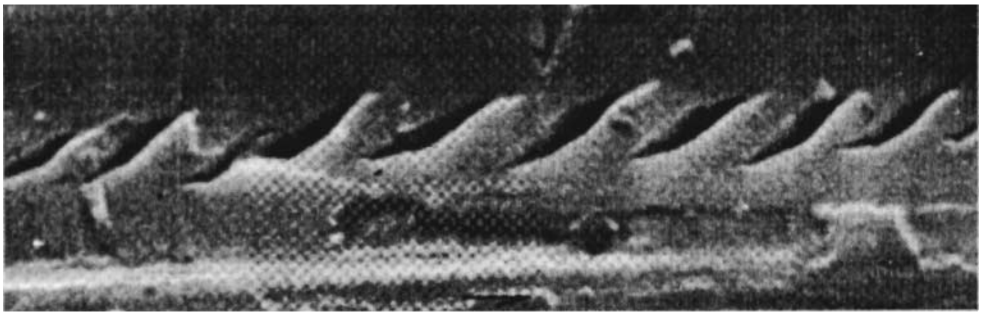

Fig. 2. Microcracks due to a shear loading (Purslow, 1981).

These definitions imply that $\vec{V}$ should require the following imperatives: to possess a single component in modes 1 and 2, and two components in a mixed mode. This implies:

$$U_T(m = 1, r) = 0 \quad \text{and} \quad U_N(m = 2, r) = 0 \quad \text{for all values of } r. \tag{4}$$

Since the variable m cannot be defined for a zero strain state, it is necessary to choose:

$$U_N(m, r = 0) = 0 \quad \text{and} \quad U_T(m, r = 0) = 0 \quad \text{for all values of } m. \tag{5}$$

We will see in the identification part that these conditions ensure the continuity of the tensor behaviour and thus the continuity of the state function.

5.3. The state function

By using a pure mesoscopic approach and then by integration on the representative volume of the material in order to obtain mesoscopic quantities. Andrieux (1983) shows that the state function can be written with two terms, the origins of which are distinct. The first one, φ, corresponds to the elastic energy stored by the equivalent material. The second one, ϕ, a function only of the internal variables describing discontinuities, corresponds to the elastic energy stored in the form of self-stress induced by discontinuities of displacement of the microcracks. Thus, we write at the mesoscopic level, the free energy function in the following form:

$$\psi = \psi(\varepsilon, \alpha, m, r) = \varphi(\varepsilon, \vec{V}(\alpha, m, r)) + \phi(m, r). \tag{6}$$

In our case, taking into account the hypothesis on the residual stresses, ϕ is equal to zero: no energy remains inside the material when the strain is equal to zero. However, for further development, this term will be kept.

The main problem in the constitutive equation formulation is to take into account the symmetries of the material. Due mainly to Pipkin and Rivlin (1959) and Boehler (1987), the standard techniques to deduce implications of symmetries consist of showing polynomial quantities which are built from chosen quantities invariant to the group of transformations respecting the symmetries of the material. In the case of orthotropic materials, the invariant quantities built with the strain tensor and the damage vector are (using Voigt' notation):

$$\varepsilon_1, \quad \varepsilon_2, \quad \varepsilon_6^2, \quad V_1^2(=V_T^2), \quad V_2^2(=V_N^2), \quad V_1 V_2 \varepsilon_6 \tag{7}$$

φ is then written as a polynomial of these invariants by combining them to obtain damageable linear elasticity. Thus, we write:

$$\varphi = \varphi^0(\varepsilon) + \varphi^N(\varepsilon, V_N) + \varphi^T(\varepsilon, V_T) + \varphi^{NT}(\varepsilon, V_n, V_T) \tag{8}$$

where:

$$\begin{cases} \varphi^0(\varepsilon) = A_1\varepsilon_1^2 + A_2\varepsilon_2^2 + A_3\varepsilon_1\varepsilon_2 + A_4\varepsilon_6^2 = F_0(\varepsilon) \\ \varphi^N(\varepsilon, V_N) = (B_1\varepsilon_1^2 + B_2\varepsilon_2^2 + B_3\varepsilon_1\varepsilon_2 + B_4\varepsilon_6^2)V_N^2 = F_N(\varepsilon) = F_N(\varepsilon)V_N^2 \\ \qquad = F_N(\varepsilon) f^2(\alpha) U_N^2(m, r) \\ \varphi^T(\varepsilon, V_T) = (C_1\varepsilon_1^2 + C_2\varepsilon_2^2 + C_3\varepsilon_1\varepsilon_2 + C_4\varepsilon_6^2)V_T^2 = F_T(\varepsilon)V_T^2 = F_T(\varepsilon) f^2(\alpha) U_T^2(m, r) \\ \varphi^{NT}(\varepsilon, V_N, V_T) = (D_1\varepsilon_1\varepsilon_6 + D_2\varepsilon_2\varepsilon_6 + D_3\varepsilon_6^2)V_N V_T = F_{NT}(\varepsilon)V_N V_T \\ \qquad = F_{NT}(\varepsilon) f^2(\alpha) U_N(m, r) U_T(m, r). \end{cases} \tag{9}$$

If we note:

$$C^0 = \begin{pmatrix} 2A_1 & A_3 & 0 \\ A_3 & 2A_2 & 0 \\ 0 & 0 & 2A_4 \end{pmatrix}, \quad C^{\mathrm{N}} = \begin{pmatrix} 2B_1 & B_3 & 0 \\ B_3 & 2B_2 & 0 \\ 0 & 0 & 2B_4 \end{pmatrix}, \quad C^{\mathrm{T}} = \begin{pmatrix} 2C_1 & C_3 & 0 \\ C_3 & 2C_2 & 0 \\ 0 & 0 & 2C_4 \end{pmatrix},$$

$$C^{\mathrm{NT}} = \begin{pmatrix} 0 & 0 & D_1 \\ 0 & 0 & D_2 \\ D_1 & D_2 & D_3 \end{pmatrix} \tag{10}$$

we can write:

$$\begin{aligned}\psi(\varepsilon,\alpha,m,r) &= 1/2[C^0_{ij}+f^2(\alpha)\{C^{\mathrm{N}}_{ij}U^2_{\mathrm{N}}(m,r)+C^{\mathrm{T}}_{ij}U^2_{\mathrm{T}}(m,r)+C^{\mathrm{NT}}_{ij}U_{\mathrm{N}}(m,r)U_{\mathrm{T}}(m,r)\}]\varepsilon_i\varepsilon_j \\ &= 1/2C_{ij}(\alpha,m,r)\varepsilon_i\varepsilon_j.\end{aligned} \tag{11}$$

The tensors C^{N}, C^{T} and C^{NT} represent the influence of the damage and the influence of its direction on the behaviour. They show precisely which components of the behaviour are sensitive to any particular direction of the damage. We observe that the geometry of the defect adds only a weak anisotropy in comparison with the initial material: due to the term C^{NT} the initial orthotropy becomes monoclinic.

5.4. State laws

Since ψ is a function of the state variables ε, α, m and r whose conjugate variables are respectively σ, A, M and R, we have:

$$\mathrm{d}\psi = \frac{\partial\psi}{\partial\varepsilon}\mathrm{d}\varepsilon+\frac{\partial\psi}{\partial\alpha}\mathrm{d}\alpha+\frac{\partial\psi}{\partial m}\mathrm{d}m+\frac{\partial\psi}{\partial r}\mathrm{d}r = \sigma\,\mathrm{d}\varepsilon+A\,\mathrm{d}\alpha+M\,\mathrm{d}m+R\,\mathrm{d}r. \tag{12}$$

By identifying term to term the two previous expressions, we obtain the state laws. Thus:

$$\begin{aligned}\text{(a)}\ \ \sigma &= \frac{\partial\psi(\varepsilon,m,r)}{\partial\varepsilon} = [C^0+f^2(\alpha)\{C^{\mathrm{N}}U^2_{\mathrm{N}}(m,r)+C^{\mathrm{T}}U^2_{\mathrm{T}}(m,r)+C^{\mathrm{NT}}U_{\mathrm{N}}(m,r)U_{\mathrm{T}}(m,r)\}]\varepsilon \\ &= C(\alpha,m,r)\varepsilon\end{aligned} \tag{13}$$

$$\begin{aligned}\text{(b)}\ \ A(\varepsilon,\alpha,m,r) &= \frac{\partial\psi(\varepsilon,\alpha,m,r)}{\partial\alpha} = \frac{1}{2}\frac{\partial C_{ij}(\alpha,m,r)}{\partial\alpha}\varepsilon_i\varepsilon_j \\ &= f^{(\alpha)}f'(\alpha)\{C^{\mathrm{N}}_{ij}U^2_{\mathrm{N}}(m,r)+C^{\mathrm{T}}_{ij}U^2_{\mathrm{T}}(m,r)+C^{\mathrm{NT}}_{ij}U_{\mathrm{N}}(m,r)U_{\mathrm{T}}(m,r)\varepsilon_i\varepsilon_j\end{aligned} \tag{14}$$

$$\begin{aligned}\text{(c)}\ \ M(\varepsilon,\alpha,m,r) &= \frac{\partial\psi(\varepsilon,\alpha,m,r)}{\partial m} \\ &= f^2(\alpha)\Bigg[F_{\mathrm{N}}(\varepsilon)\cdot 2U_{\mathrm{N}}(m,r)\frac{\partial U_{\mathrm{N}}(m,r)}{\partial m}+F_{\mathrm{T}}(\varepsilon)\cdot 2U_{\mathrm{T}}(m,r)\frac{\partial U_{\mathrm{T}}(m,r)}{\partial m} \\ &\quad +F_{\mathrm{NT}}(\varepsilon)\cdot\left\{U_{\mathrm{T}}(m,r)\frac{\partial U_{\mathrm{N}}(m,r)}{\partial m}+U_{\mathrm{N}}(m,r)\frac{\partial U_{\mathrm{T}}(m,r)}{\partial m}\right\}\Bigg]+\frac{\partial\phi(m,r)}{\partial m}\end{aligned} \tag{15}$$

$$\text{(d)}\quad R(\varepsilon,\alpha,m,r) = \frac{\partial\psi(\varepsilon,\alpha,m,r)}{\partial r}$$
$$= f^2(\alpha)\left[F_N(\varepsilon)\cdot 2U_N(m,r)\frac{\partial U_N(m,r)}{\partial r} + F_T(\varepsilon)\cdot 2U_T(m,r)\frac{\partial U_T(m,r)}{\partial r}\right.$$
$$\left. + F_{NT}(\varepsilon)\cdot\left\{U_T(m,r)\frac{\partial U_N(m,r)}{\partial r} + U_N(m,r)\frac{\partial U_T(m,r)}{\partial r}\right\}\right] + \frac{\partial\phi(m,r)}{\partial r}. \tag{16}$$

5.5. Evolution laws

Using the free-energy, we get the Clausius–Duhem inequality:

$$\sigma\dot{\varepsilon} - \dot{\psi} \geqslant 0 \quad \text{for all evolution of the system.} \tag{17}$$

Thus, any restriction is imposed to the state variables except for damage which describes an irreversible phenomenon. Then: $\dot{\alpha} \geqslant 0$. In consequence, taking into account the state laws, we get:

$$-A(\varepsilon,\alpha,m,r)\dot{\alpha} - M(\varepsilon,\alpha,m,r)\dot{m} - R(\varepsilon,\alpha,m,r)\dot{r} \geqslant 0 \quad \forall\dot{\alpha} \geqslant 0,\quad \forall\dot{m},\quad \forall\dot{r}. \tag{18}$$

Since we consider the case where the microcracks are opened or the friction between their lips is non-existent, the variables m and r are non dissipative. That means:

$$M = 0 \text{ and } \quad R = 0. \tag{19}$$

Finally, the inequality of Clausius–Duhem reduces to:

$$-A(\varepsilon,\alpha,m,r)\dot{\alpha} \geqslant 0 \quad \forall\dot{\alpha} \geqslant 0. \tag{20}$$

5.5.1. Damage evolution law

According to eqn (20), we write the damage development law with a convex criterion $c(\varepsilon,\alpha,m,r)$ by using a damage threshold and the consistency hypothesis. We choose:

$$c(\varepsilon,\alpha,m,r) = A^c(\alpha,m,r) - A(\varepsilon,\alpha,m,r) \leqslant 0 \tag{21}$$

where $A^c(\alpha,m,r)$ is the damage threshold. Setting $c(\varepsilon,\alpha,m,r) = 0$ and $\mathrm{d}c(\varepsilon,\alpha,m,r) = 0$ during damaging, we get:

$$\mathrm{d}\alpha = -\frac{\dfrac{\partial^2\psi(\varepsilon,\alpha,m,r)}{\partial\alpha\partial\varepsilon_i}\mathrm{d}\varepsilon_i + \left(\dfrac{\partial^2\psi(\varepsilon,\alpha,m,r)}{\partial\alpha\partial m} - \dfrac{\partial A^c(\alpha,m,r)}{\partial m}\right)\mathrm{d}m + \left(\dfrac{\partial^2\psi(\varepsilon,\alpha,m,r)}{\partial\alpha\partial r} - \dfrac{\partial A^c\psi(\alpha,m,r)}{\partial r}\right)\mathrm{d}r}{\dfrac{\partial^2\psi(\varepsilon,\alpha,m,r)}{\partial\alpha^2} - \dfrac{\partial A^c(\alpha,m,r)}{\partial\alpha}} \tag{22}$$

Comparing with former studies, the dependence of the damage threshold versus the variables m and r takes into account the fact that the necessary energy for the creation of a crack is different according to the loading mode.

5.5.2. Consequences of the non-dissipative character of the variables m and r

As a first step, using eqns (15) and (16) and the non-dissipative character of m and r, eqn (19), we can state from the implicit functions theorem that (locally): $m = m(\varepsilon, \alpha)$ and $r = r(\varepsilon, \alpha)$. If α is constant, we find as already demonstrated by Andrieux (1983) that the internal variables describing the discontinuities due to the presence of the microcracks are functions of the mesoscopic state of strain.

6. Identifications

6.1. Identification of m and r

The goal of this section is to find an analytic expression $m(\varepsilon, \alpha)$ and $r(\varepsilon, \alpha)$. To simplify, we assume that the strain ε_1 has no influence on the appearance of the damage. Finally, that means:

- the yield between traction and compression of microcrack lips is $\varepsilon_2 = 0$;
- the failure by the opening of the microcrack lips (mode 1) is exclusively due to ε_2;
- the failure by the sliding of the microcrack lips (mode 2) is exclusively due to ε_6.

By taking inspiration from a form of a failure criterion, where the quantities $\varepsilon_2^c(\alpha)$ and $\varepsilon_6^c(\alpha)$ are the failure strain of the equivalent material, eventually depending on the damage and $k(\alpha) = \varepsilon_2^c(\alpha)/\varepsilon_6^c(\alpha)$, the variables m and r are defined with the following form:

$$\text{if } \varepsilon_2 \geqslant 0: \quad m(\varepsilon, \alpha) = \frac{2\varepsilon_6^2 + \dfrac{\varepsilon_2^2}{k^2(\alpha)}}{\varepsilon_6^2 + \dfrac{\varepsilon_2^2}{k^2(\alpha)}} \quad \text{and} \quad r(\varepsilon, \alpha) = r(\varepsilon) = \sqrt{\varepsilon_2^2 + \varepsilon_6^2}$$

$$\text{if } \varepsilon_2 < 0: \quad m(\varepsilon, \alpha) = 2 \quad \text{and} \quad r(\varepsilon, \alpha) = r(\varepsilon) = \sqrt{\varepsilon_6^2}. \tag{23}$$

The value $m = 2$ means that the microcracks are loaded in pure shear mode, i.e. their lips are closed. This is also the case when they are in compression. Therefore, $m = 2$ characterizes, at the mesoscopic level, a microcracks network under shear ($r \neq 0$), compression ($r = 0$) or simultaneously shear and compression ($r \neq 0$). We observe:

- if $\varepsilon_2 \neq 0$ and $\varepsilon_6 = 0$ then $m(\varepsilon, \alpha) = 1 \; \forall \; \varepsilon_2 \mathrm{k}(\alpha)$;
- if $\varepsilon_2 = 0$ and $\varepsilon_6 \neq 0$ then $m(\varepsilon, \alpha) = 2 \; \forall \; \varepsilon_6 \; \forall \; k(\alpha)$;
- if $\varepsilon_2 = \varepsilon_6$ with $k(\alpha) < 1$ then $m(\varepsilon, \alpha) < 1.5$ (if $k(\alpha) \to 0$, $m(\varepsilon, \alpha) \to 1$). This indicates that a material more brittle in tension than in shear, submitted to equal loadings along these two directions, is going to fail in tension: the surfaces of the microcracks move apart and are loaded in a mode close to 1;
- $\varepsilon_2 = \varepsilon_6$ with $k(\alpha) > 1$, then $m(\varepsilon, \alpha) > 1.5$ (if $k(\alpha) \to +\infty$, $m(\varepsilon, \alpha) \to 2$). This situation is opposite to the above case, i.e. the surfaces of the microcracks are sheared and therefore loaded in a mode close to 2;
- $\varepsilon_2 = \varepsilon_6$ with $k(\alpha) = 1$, then $m(\varepsilon, \alpha) = 1.5$. This is the limit case of the two preceeding cases where material is isotropic: therefore it is normal to find the median value of 1 and 2.

These above remarks confirm the legitimacy of the choice of the expressions of the function m and r.

6.2. Numerical identification of the behaviour tensor

The tensor C^N is the stiffness loss due to the component V_N of the damage vector, when the microcracks are opened but not slid: thus $B_4 = 0$. The tensor C^T is the stiffness loss due to the component V^T of the damage vector, when the microcracks are submitted to a shear but are not opened: thus $C_1 = C_2 = C_3 = 0$. The behaviour tensor is written therefore:

$$C = C^0 + f^2(\alpha) \begin{bmatrix} 2B_1 U_N^2(m,r) & B_3 U_N^2(m,r) & D_1 U_N(m,r) U_T(m,r) \\ B_3 U_N^2(m,r) & 2B_2 U_N^2(m,r) & D_2 U_N(m,r) U_T(m,r) \\ D_1 U_N(m,r) U_T(m,r) & D_2 U_N(m,r) U_T(m,r) & 2C_4 U_T^2(m,r) \\ & & +2D_3 U_N(m,r) U_T(m,r) \end{bmatrix}. \quad (24)$$

When $m = 2$, this allows us to effectively take into account the unilateral effect of the damage. We get:

$$C = C^0 + f^2(\alpha) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2C_4 U_T^2(m = 2, r) \end{pmatrix}. \quad (25)$$

As the variables m and r are now defined, an important remark concerning the behaviour tensor can be made. The cracked medium homogenization (Leguillon and Sanchez—Palencia, 1982) shows:

- that the behaviour of the equivalent material has to depend on the direction of the stress tensor. The dependence of this magnitude appears here with m;
- that the behaviour of the equivalent material does not depend on the amplitude of the stress. However a dependence of this magnitude appears here with r.

In fact, determining the cracks are opened or closed has no physical meaning. Indeed, local roughness or scraps that can fill the space between the crack lips induce a progressive restoration of the properties of the material. Especially, the tangent behaviour has to be continuous from tension to compression. This variable r allows this condition to be satisfied. In conclusion, the effect of r is important when ε_2 reaches 0. Other, it has to be weakened. For instance, we can choose:

$$U_N(m,r) = h_N(m) j_N(r);$$

$$U_T(m,r) = h_T(m) j_T(r)$$

$$j_N(r) = j_T(r) = j(r) = 1 - e^{-xr}, \quad \text{with } x \text{ a large positive real.} \quad (26)$$

Thus, we find that except for the vicinity of the point of zero strains, the behaviour depends on the orientation of the stress tensor, i.e. on m, and not too much on r. Therefore, we can write:

$$C = C^0 + (1-e^{-xr})^2 f^2(\alpha) \begin{pmatrix} 2B_1 h_N^2(m) & B_3 h_N^2(m) & D_1 h_N(m) h_T(m) \\ B_3 h_N^2(m) & 2B_2 h_N^2(m) & D_2 h_N(m) h_T(m) \\ D_1 h_N(m) h_T(m) & D_2 h_N(m) h_T(m) & 2C_4 h_T^2(m) + D_3 h_N(m) h_T(m) \end{pmatrix}. \tag{27}$$

Finally, the complete identification of the functions $h_T(m)$, $h_N(m)$ and $f(\alpha)$ and of the coefficient B_1, B_2, B_3, C_4, D_1, D_2 and D_3 is realized by smoothing the results of homogenization calculations on a representative cell of the damaged material (Thionnet and Renard, 1993).

This homogenization step allows us also to show that the effect of a microcrack is very localized. In particular, the presence of the crack does not affect significantly the state of stress of the neighbouring plies. This important point justifies completely the replacement of the damaged material by an equivalent homogeneous one. At least, it is necessary to point out that, although m is a discontinued function, the effect of r, eqns (5) and (26), allows the continuity of the tensor C, eqn (27), and then of the free energy.

6.3. Identification of the damage threshold

The identification process of the threshold $A^c(\alpha)$ is identical to former studies (Thionnet and Renard, 1993). It is necessary to get the experimental curve of crack density versus the applied stress on a well chosen laminate. If we wish to take into account that the necessary energy to create a crack depends on the loading mode, we need experimental information where the crack density is given for plies submitted to different values of m (for examples on sequences $[0^\circ, 90^\circ]_s$, $[0^\circ, \pm 30^\circ]_s$, $[0^\circ, \pm 45^\circ]_s$ and $[0^\circ, \pm 60^\circ]_s$). Then, by an inverse procedure, by giving the experimental crack density evolution, we calculate the variable A and we write that during the damage process $A = A^c$. A fitting of these results provides the function $A^c(\alpha, m)$.

7. Schematic functioning of the model

We wish now to give a qualitative overview of the model. For this reason, the units of the axes are voluntarily left out. The behavior of the model is studied by considering responses which may arise. Thus, we choose $h_N(m) = 2-m$ and $h_T(m) = m-1$, $f(\alpha) = \alpha^{1/8}$ and $D_1 = D_2 = 0$ and the expression of the threshold A^c (i.e. the damage evolution law).

We fixed the path of strain (Fig. 3a) to be applied to the element of the representative volume of the material. Figure 3b shows the induced evolutions of r and m. Finally, Fig. 3c gives responses of the behaviour law in terms of stress/strain curves.

Between times $t = 0$ and $t = 5$, we impose a positive increasing strain ε_2. The microcracks are loaded in mode 1 (opened lips: $m = 1$). The responses σ_1/ε_2 and σ_2/ε_2 are non-linear because of the growth of the damage.

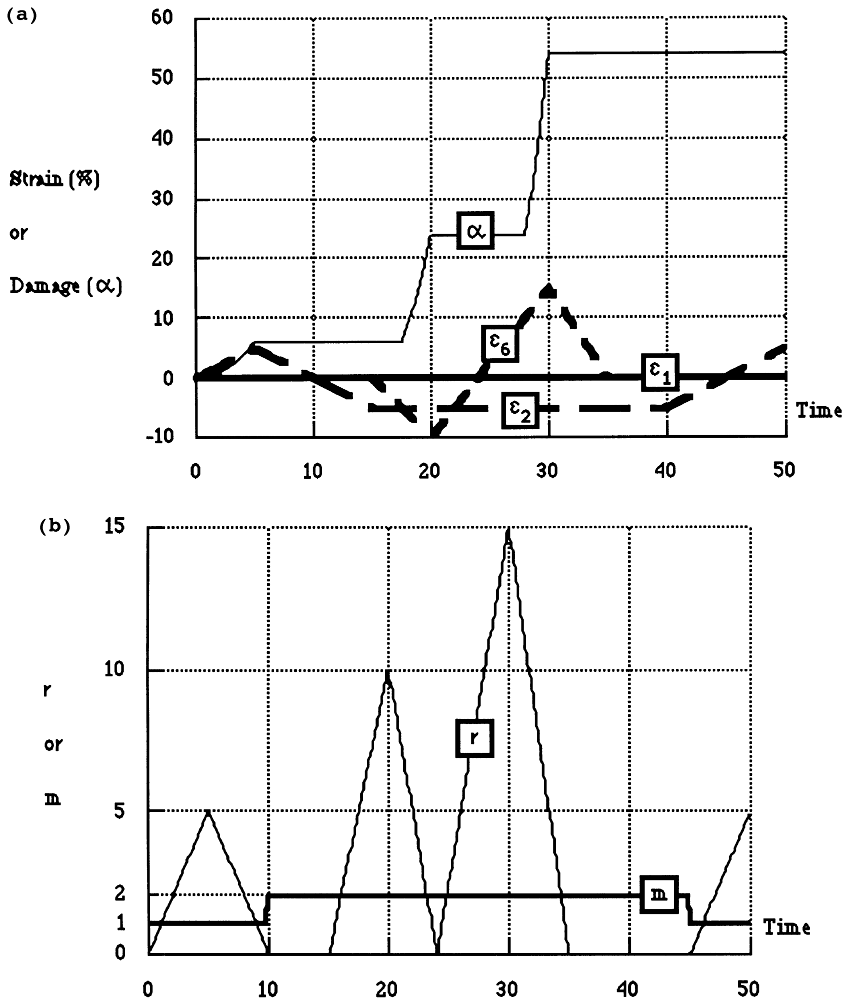

Fig. 3. (a) Applied strains and damage evolution. (b) Evolutions of m and r.

Between $t = 5$ and $t = 15$, we impose a decreasing strain ε_2. The responses σ_1/ε_2 and σ_2/ε_2 are linear (damageable elasticity). When ε_2 becomes negative (from $t = 10$), the microcracks are closed ($m = 2$) and the material recovers its initial properties.

Between $t = 15$ and $t = 20$, we impose a negative decreasing strain ε_6 (ε_2 remains constant). The damage threshold is not immediately reached but only from $t = 17.5$. From this moment, the response σ_6/ε_6 is linear (with a degraded modulus with respect to the initial material, since a damage has already been created by applying of the preceeding strain) and non-linear after.

Between $t = 20$ and $t = 30$, we impose an increasing strain ε_6 (ε_2 remains constant). The damage threshold being reached only from $t = 28$, the response σ_6/ε_6 is linear until this time. Then, the damage increases, and the response σ_6/ε_6 becomes non-linear. When $\varepsilon_6 = 0$ the value of the shear modulus is different from its initial value because the unilateral condition of damage does not affect this modulus.

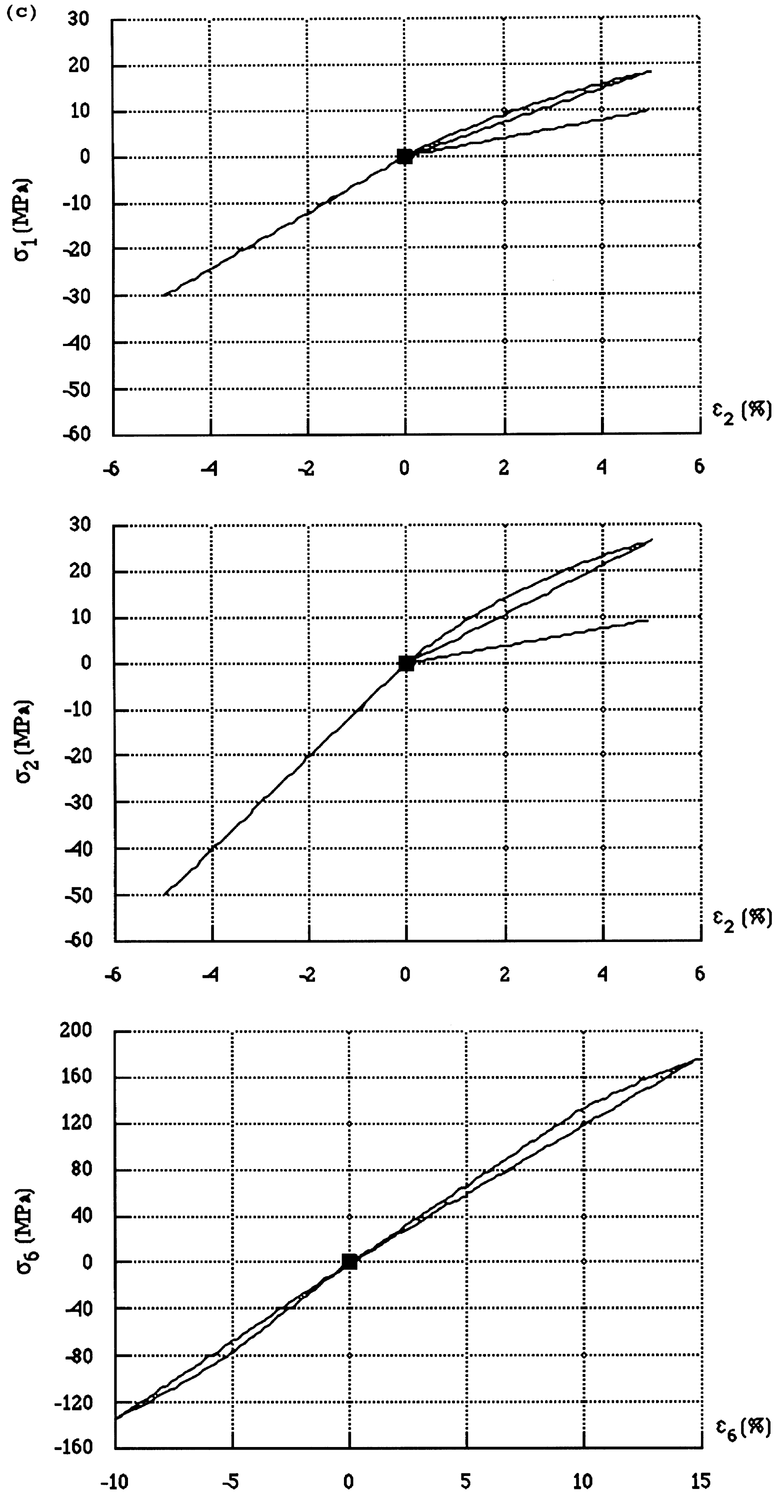

Fig. 3. (c) Stress/strain response.

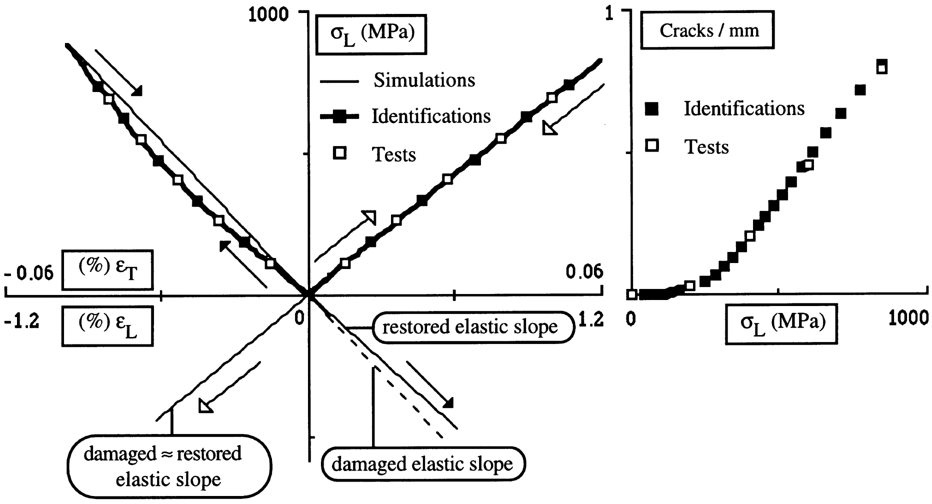

Fig. 4. Simulation on the $[0^\circ_2, 90^\circ_2]_s$ stacking sequence.

Between $t = 30$ and $t = 40$, the strain ε_6 decreases to zero. The damage is constant and the response σ_6/ε_6 is linear.

Between $t = 40$ and $t = 50$, we increase the strain ε_2. The microcracks are loaded in mode 2 and then the fracture surfaces are closed ($m = 2$), because the strain is not positive (until $t = 45$). Afterward, they are loaded in mode 1 ($m = 1$). Even if the damage is now constant, since it has evolved previously, the responses σ_1/ε_2 and σ_2/ε_2 are linear but show a weaker modulus than at the end of the first application of ε_2.

8. Application to a composite material

We propose an application of the model to a carbon/epoxy composite. The experimentally and numerically tested volume element is a thin plane plate. The loading is uniaxial and applied along the axis of the specimen. Two stacking sequences $[0^\circ_2, 90^\circ_2]_s$ and $[60^\circ_2, 90^\circ_2]_s$ are used. The first one is also used for the identifications of the model. For each of the plates (Figs 4–5), we calculate:

- the responses $\varepsilon_T - \varepsilon_L/\sigma_L$ of the specimen (ε_T: transverse strain, ε_L: longitudinal strain, σ_L: applied longitudinal stress);
- the evolution of the microcrack density within the plies versus σ_L.

Unfortunately, the authors could not find tests where simultaneously:

- the material and the geometry of the specimen are in the framework of the study;
- the transverse and longitudinal responses are measured;
- a compression loading is realized.

The reason is that for this type of structure, buckling appears very fast during the compression.

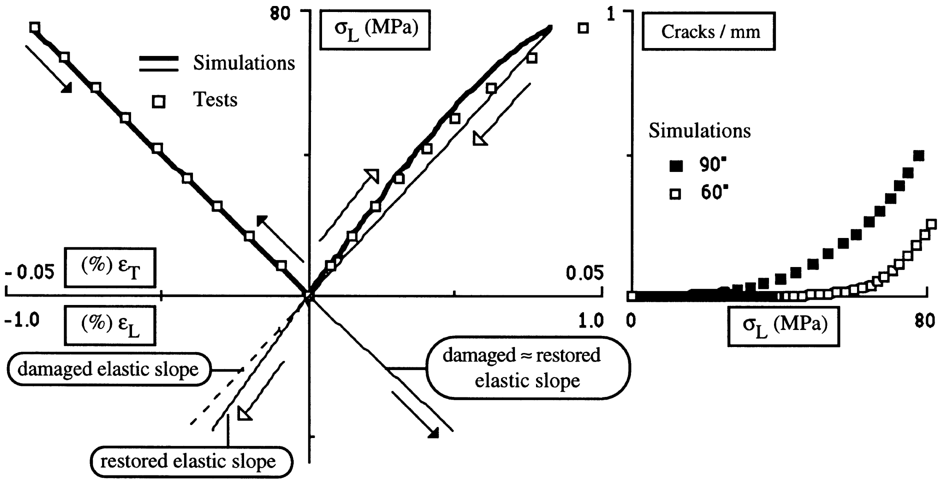

Fig. 5. Simulation on the $[60^\circ_2, 90^\circ_2]_s$ stacking sequence.

Simulations cannot therefore be compared to experiments in the compression domain. We compare our numerical results to the experimental tests made by Jeggy (1990).

Concerning the $[0^\circ_2, 90^\circ_2]_s$ stacking sequence:

- the influence of damage on the response ε_L/σ_L is not significant. This allows us to explain why we could not distinguish between the damaged and the restored slopes (Fig. 4);
- the influence of the damage on the response ε_T/σ_L is substantial. We clearly observe the difference between the damaged and the restored slopes (Fig. 4) which is exactly equal to the initial slope. This is linked to the compression path which in the case of the 90° ply coincides with the complete closing of all the microcracks.

Concerning the $[60^\circ_2, 90^\circ_2]_s$ stacking sequence:

- the influence of the damage on the response ε_T/σ_L is not significant. This allows us to explain why we could not distinguish the damaged and the restored slopes (Fig. 5);
- the influence of the damage on the response ε_L/σ_L is substantial. We observe clearly the difference between the damaged and the restored slopes (Fig. 5). However, the restored slope is not exactly equal to the initial slope. This is due to the fact that the initial shear modulus of the 60° ply is never recovered and contributes to the global rigidity of the laminate. The restored slope is therefore weaker than the initial slope. However this difference is not important on the figure.

9. Conclusion

We propose a model which takes into account the unilateral character of the damage linked to the application of compression loading on a microcracked material.

The originality of the study is first to use two internal non dissipative variables (m and r) that characterize the displacement discontinuities of the surfaces of the microcracks at the mesoscopic level. Then, these two variables allow us to write a single state function for compression and tension states. The symmetry of the behaviour tensor is then established. It is the same for the continuity of the stress/strain responses.

The two variables m and r define the loading mode in damage mechanics like in fracture mechanics and allows to take into account the unilateral character of the damage. Among all the hypotheses, the most important one supposes that the defects develop into preferential directions. This limits the additional anisotropy induced by the damage. Thus, this model can be essentially applied to strongly anisotropic materials.

References

Allen, D.H., Harris, C.E., Groves, S. E., 1987. A thermomechanical constitutive theory for elastic composites with distributed damage, Parts I and II. International Journal of Solids and Structures 23 (9), 1301–1318.

S. Andrieux., 1983. Un modèle de matériau microfissuré. Applications aux roches et aux bétons. Thèse, Université Paris VI, France.

Aussedat, E., Thionnet, A., Renard, J., 1995. Comportement en compression des composites par une définition du mode de sollicitation en mécanique de l'endommagement. Compte Rendu de l'Académie des Sciences de Paris série IIb, 321, 533–540.

Boehler, J.P., 1987. Applications of Tensor Functions in Solids Mechanics. Springer–Verlag.

Chaboche, J.L., 1992. Une nouvelle condition unilatérale pour décrire le comportement des matériaux avec dommage anisotrope. Compte Rendu de l'Académie des Sciences de Paris, série II, 314, 1395–1401.

Highsmith, A.L., Reifsnider, K.L., 1982. Stiffness reduction mechanisms in composite laminate. In: Reifsnider, K.L. Ed.), Damage in Composite Materials. ASTM STP 775, American Society for Testing and Materials, pp. 103–117.

Jeggy, T., 1990. Modélisation des dégradations des plaques trouées en carbone/époxy sous chargements biaxiaux quasi-statiques, Thèse, Ecole Centrale de Paris, France.

Krajcinovic, D., Fonseka, G.U., 1981. The continuous damage theory of brittle materials, part 1: general theory. Journal of Applied Mechanics 48, 809–824.

La Borderie, C., Pijaudier-Cabot, G., Mazard, J., 1992. Computational modelling of concrete and reinforced concrete based on damage mechanics. Proceedings of Third International Conference of Computational Plasticity, p. 681–692.

Lee, J.W., Harris, C.E., Allen, D.H., 1989. Internal state variable approach for predicting stiffness reduction in fibrous laminated composites with matrix cracks. Journal of Composite Materials 23, 1273–1291.

Leguillon, D., Sanchez-Palencia, E., 1982. On the behavior of a cracked elastic body with or without friction. Journal de Mécanique Théorique et Appliquée 1 (2), 195–209.

Nuismer, R.J., 1980. Predicting the performance and failure of multidirectional polymeric matrix composite laminates: a combined micro-macro approach. proceedings of the Third International Conference on Composite Materials, vol. 1, Paris, France, p. 436–452.

Pipkin, A.C., Rivlin, R.S., 1959. The formulation of constitutive equations in continuum physics. Archives of Rational Mechanics Analysis 5, 129–144.

Purslow, D., 1981. Some fundamental aspects of composites fractography. Composites 12, October, 241–247.

Purslow, D., 1986. Matrix fractography of fibre-reinforced epoxy composites. Composites 17, October, 289–303.

Ramtani, S., 1990. Contribution à la modélisation du comportement multiaxial du béton endommagé avec description du caractère unilatéral. Thèse, Paris VI, France.

Reifsnider, K.L., 1977. Some fundamental aspects of fatique and fracture response of composite materials. Proceedings of 14th Meeting of Society of Engineering Science, Bethlehem, pp. 373–384.

Renard, J., Favre, J.P.. Jeggy, T., 1993. Influence of transverse cracking on ply behavior: introduction of a characteristic damage variable. Composites Science and Technology 46, 29–37.

Talreja, R., 1985. Transverse cracking and stiffness reduction in composite laminates. Journal of Composites Materials 19, 355–375.

Thionnet, A., Renard, J., 1993. Meso-macro approach to transverse cracking in laminated composites using Talreja's model. Composites Engineering 3 (9), 851–871.